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Council is another work of which Mr. Miller has charge, and in noticing this the *Medical Press* says—"We assume that its compilation is chiefly due to the energy and noted mathematical skill of Mr. Miller, the Registrar of the Council, and if we are correct in this assumption we can only remark that both the profession and the Medical Council owe that gentleman much thanks for work which, though no doubt a labour of love, must involve great devotion of time and mental capacity." Another work of the utmost importance to the public, and for the annual publication of which Mr. Miller is responsible, is the *Medical Register*, which has now grown to a volume of 1,198 pages. In addition to this there is the *Dentist's Register* (232 pages), besides the *Medical Students' Register*, the latter alone requiring 100 pages. It would be difficult to speak too highly of the care exhibited in the compilation of these important works. Referring to the issues for the present year, the *Medical Press* says "They display all the progressive improvement which has been manifested since Mr. Miller took them in hand."

Being an ardent lover of science and literature, Mr. Miller has all his life striven to aid others in sharing their delights, by lectures, writings, and teaching. And all this work, editorial and other, has been not only unremunerative, but carried on with no little outlay. But the world's best workers have always been the most unselfish. Mr. Miller has at least the gratification of knowing that his favourite pursuits have been greatly advanced by his efforts, and that he has earned the gratitude of many who have reaped the advantages of his self-denying work."

Mr. Miller was one of the earliest members of the London Mathematical Society; but as he found that, with his official duties, and his Editorial work, he could not spare time to attend the Meetings, he was reluctantly compelled to resign his membership. Since that time, he has had to devote the whole of his small leisure to the duties of his Editorship, which goes on increasing every month, with new contributors from foreign countries, especially India, where an enlarged interest is rapidly growing in all the articles that are published in his Journal. Mr. Miller is a great admirer of America and American ways of managing; he entertains a high opinion of our magazine, and says it is one of the best that comes to him. He has a large circle of friends and admirers in America, most of whom are contributors to the Mathematical Department in the *Educational Times*.

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THE EXPONENTAL DEVELOPMENT FOR REAL EXPONENTS.

By WILLIAM BENJAMIN SMITH. Ph. D., (Görtingen) Professor of Mathematics, Tulane University, New Orleans, Louisiana.

The Exponental Series is of such fundamental and far-reaching importance, it is so indispensable to all higher Analysis, that it seems strange so few if any deductions of it accessible to the English reader should be carefully conducted; not even that given by Chrystal in his superb *Treatise on Algebra* can lay claim to rigor. It may be worth while then, under no pretense of novelty, to attempt to supply this lack in some measure.

I. We consider the expansion given by the Binomial Theorem:

$$\left(1+\frac{x}{n}\right)^n=1+n.\frac{x}{n}+\frac{n(n-1)}{\underline{|2}}.\frac{x^2}{n^2}+\frac{n(n-1)(n-2)}{\underline{|3}}.\frac{x^3}{n^3}+\ldots\ldots$$

$$=1+x+(1-\frac{1}{n}).\frac{x^2}{\underline{|2}}+(1-\frac{1}{n})(1-\frac{2}{n}).\frac{x^3}{\underline{|3}}+\ldots\ldots,$$

where x is finite and positive, while n is *positive* and *integral*, and we inquire whether this series tends toward a definite form and value as n increases without limit.

We denote the differences $1-\frac{1}{n}$, $1-\frac{2}{n}$, …… $1-\frac{k}{n}$ …… by d_1, d_2, ……d_k, …… and the products of these, d_1, d_1d_2, $d_1d_2d_3$, ……d_1d_2 ……d_k, ……by p_1, p_2, p_3, ……p_k, ……

Then plainly $1>d_1>d_2>d_3>\ldots\ldots>d_k>\ldots\ldots$

and also $1>p_1>p_2>p_3>\ldots\ldots>p_k>\ldots\ldots$

In the expansion there are $n+1$ terms, which we may write t_0, t_1, t_2, ……t_r, ……t_n. We consider the sum $t_0+t_1+\ldots\ldots+t_r$ and denote it by S_r; then the sum of remaining $n-r$ terms we denote by V_r, so that $\left(1+\frac{x}{n}\right)^n=S_r+V_r$ where

$$S_r=1+x+p_1\frac{x^2}{\underline{|2}}+p_2\frac{x^3}{\underline{|3}}+\ldots\ldots+p_k.\frac{x^{k+1}}{\underline{|k+1}}+\ldots\ldots+p_{r-1}.\frac{x^r}{\underline{|r}},$$

$$V_r=p_r.\frac{x^{r+1}}{\underline{|r+1}}+p_{r+1}.\frac{x^{r+2}}{\underline{|r+2}}+\ldots\ldots+p_{n-1}.\frac{x^n}{\underline{|n}}.$$

Since n is to be taken *great at will*, r may also be taken *great at will* and yet *always less than* n. We now ask, what becomes of S_r as r increases without limit while always $r<n$? Since all the p's are <1, it is plain that

$$S_r<\{1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots+\frac{x^r}{\underline{|r}}\}.$$

However we can make *each* of the p's$>1-\sigma$, where σ is *small at will*. It is enough to prove this for the *least* of the p's, p_r. We have

$$p_r=(1-\frac{1}{n})(1-\frac{2}{n})\ldots\ldots(1-\frac{r-1}{n})>(1-\frac{r-1}{n})^{r-1}.$$

Now $(1-\frac{r-1}{n})^{r-1}=1-\sigma$ if $1-\frac{r-1}{n}=(1-\sigma)^{\frac{1}{r-1}}$, or if $\frac{r-1}{n}=1-(1-\sigma)^{\frac{1}{r-1}}$, or if

$$n \geqq \frac{r-1}{1-(1-\sigma)^{\frac{1}{r-1}}}.$$

Now for any finite value of r *however great*, and for any finite value of σ *however small*, this fraction on the right will always be finite and perfectly difinite, though perhaps very great; hence it will always be possible to choose n equal or even greater (in case the fraction be not integral in value); hence it will always be possible to make $p_r > 1-\sigma$, no matter how great r or how small σ, by merely choosing n great enough, and this we can always do, since n is quite at our will.

$$\text{Hence } S_r > (1-\sigma)\{(1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

$$\text{Hence } \{1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\} > S_r > (1-\sigma)\ \{1+x+\frac{x^2}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

Hence S_r differs from $\{\ldots.\}$ by less than $\sigma\{\ldots.\}$. Now this brace, $\{\ldots.\}$ is of course finite for all finite values of r, and it also remains finite (for x finite) even for r increasing without limit. For the ratio of two consecutive terms is $\frac{x}{k}$ and this ratio is not only finitely <1 for $k > x$ but it becomes ever smaller and smaller, sinking below every assignable degree of parvitude as k increases without limit, x of course being finite and fixed, however great. Hence $\sigma\{\ldots\ldots\}$ is small at will, since the product of a magnitude small at will multiplied by a finite number is itself small at will. Hence the two magnitudes $\{\ \ldots..\}$ and $(1-\sigma)\{\ldots..\ \}$ close down upon each other as r increases without limit, hence they close down upon S_r always between them, so that we have

$$\text{Lim.} S_r = 1+x+\frac{x^r}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}.$$

for r and n increasing without limit, $n > r$. It remains to examine V_r. We may write $V_r = t_{r+1}(1+d_{r+1}\,\frac{x}{r+2}+d_{r+1}d_{r+2}\,\frac{x^2}{(r+2)(r+3)}+ \ldots..)$.

Now $t_{r+1} < \frac{x^{r+1}}{\underline{|r+1}}$ and this we have just seen is small at will for r great at will, or $t_{r+1} < \sigma$. Also the parenthesis $(\ \ldots.\ ) < [1+\frac{x}{r+2}+\frac{x^2}{(r+2)^2}+\ldots.]$,

and this bracket $[\ldots\ldots]$ is finite for all finite values of n and r, and it has 1 for its limit as n and r increase without limit. Hence $V_r < \sigma$, or Lim. $V_r = 0$, hence

$$\text{Lim.}\,(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots \textit{ in infinitum.}$$

II. Thus far n has been integral at every stage of value. What if it be *fractional* or *irrational?* The preceding proof does not then apply but we shall always have n lying between two consecutive integers, or $w < n < w+1$.

Now $(1+\frac{x}{n})^n < (1+\frac{x}{w})^{w+1}$, a greater number raised to a higher power; or $(1+\frac{x}{n})^n < (1+\frac{x}{w})^w\,(1+\frac{x}{w})$. Likewise $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^w$, a smaller number to a lower power. Or $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^{w+1} / (1+\frac{x}{w+1})$.

Now for w and $w+1$ increasing without limit we have just proved that $(1+\frac{x}{w})^w$ and $(1+\frac{x}{w+1})^{w+1}$ both close down upon one and the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

Also the multiplier $1+\frac{x}{w}$ and the divisor $1+\frac{x}{w+1}$ both close down upon the same limit 1; hence the product $(1+\frac{x}{w})^w.(1+\frac{x}{w})$ and the quotient $(1+\frac{x}{w+1})^{w+1}/(1+\frac{x}{w+1})$ both close down upon the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots;$$

hence $(1+\frac{x}{n})^n$ lying always between this product and this quotient itself closes down upon the same limit; hence $\text{Lim.}(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots\ldots$ for finite positive x and for positive n increasing no matter how without limit.

III. For x *negative* we must consider $(1-\frac{x}{n})^n$. This we write $=E-O$, a sum of even powers, less a sum of odd powers, of x. Each of these we break up into two parts, S_r and V_r, and reiterate the foregoing argument with insignificant and self-evident modifications. There results

$$\text{Lim.}(1-\frac{x}{n})^n = 1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\ldots\ldots$$

for finite positive x and n increasing no matter how without limit. Or

$$\text{Lim.}(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

for any finite real x positive or negative, n increasing any way without limit.

IV. For n *negative* we have

$$\left(1-\frac{x}{n}\right)^{-n}=\left(\frac{n-x}{n}\right)^{-n}=\left(\frac{n}{n-x}\right)^{n}=\left(1+\frac{x}{n-x}\right)^{n}=$$

$$\left(1+\frac{x}{n-x}\right)^{n-x}\left(1+\frac{x}{n-x}\right)^{x}=\left(1+\frac{x}{n-x}\right)^{n-x}\Big/\left(1-\frac{x}{n}\right)^{x}.$$

Now as n increases without limit, so also does $n-x$, for x finite no matter how great; hence $(1+\frac{x}{n-x})^{n-x}$ approaches as its limit the series $1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ and $(1-\frac{x}{n})^x$ approaches 1 as its limit manifestly; hence

$$\text{Lim.}(1-\frac{x}{n})^{-n} = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

Hence $\text{Lim.}(1+\frac{x}{n})^n=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ *for all finite real values of x, for real n increasing without limit no matter how, positively or negatively.*

For $x=1$ we obtain $\text{Lim.}(1+\frac{1}{n})^n=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\ldots.$

The number defined by this series and denoted by e, is one of the three irrationals (π, i, e) all-important to analysis. That e is irrational may be easily seen thus: Consider the first $(p+1)$ terms $1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|p}}$; the sum Σp is $\frac{w}{\underline{|p}}$ where w is some integer no matter what; the remainder

$$\frac{1}{\underline{|p+1}}+\frac{1}{\underline{|p+2}}+\ldots\ldots \text{ is } \frac{1}{\underline{|p}}\left\{\frac{1}{p+1}+\frac{1}{(p+1)(p+2)}+\ldots\ldots\right\}$$

$$< \frac{1}{\underline{|p}}\left(\frac{1}{p+1} = \frac{1}{(p+1)^2} + \ldots\ldots\right) < \frac{1}{\underline{|p}}. \quad \text{Hence} \frac{w}{\underline{|p}} < e < \frac{w+1}{\underline{|p}}.$$

Now as p increases without limit these two fractions close down upon each other and upon e always between them. It is plain that there is no fixed fraction as N/D, always between $\frac{w}{\underline{|p}}$ and $\frac{w+1}{\underline{|p}}$; for however great D might be, we could choose p so large, that $\underline{|p}$ would include all the factors of D; hence $\frac{N}{D} = \frac{w}{\underline{|p}}$, whereas $e > \frac{w}{\underline{|p}}$. In fact as the two fractions $\frac{w}{\underline{|p}}$ and $\frac{w+1}{\underline{|p}}$ close down on each other and on their common limit e they *pass over* (either the one or the other) every assignable fraction lying between them.

The importance of e lies in the fact that the series $1 + x + \frac{x^2}{\underline{|2}} + \ldots\ldots.$ is expressible through it. We have

$$(1 + \frac{x}{n})^n = \{(1 + \frac{x}{n})^{n/x}\}^x = \{(1 + \frac{1}{n/x})^{n/x}\}^x. \quad \text{Hence}$$

$$\text{Lim.}(1 + \frac{x}{n})^n = \text{Lim.}\{(1 + \frac{1}{n/x})^{n/x}\}^x = \{\text{Lim.}(1 + \frac{1}{n/x})^{n/x}\}^x,$$

where we indeed assume that the Limit of the Power equals the Power of the Limit. But this is plainly correct, at least in the present case ; for

$$(1 + \frac{1}{n/x})^{n/x} = e + \sigma.$$

Hence $\text{Lim.}\{(1 + \frac{1}{n/x})^{n/x}\}^x = \text{Lim.}(e + \sigma)^x = \text{Lim.}(e^x + \sigma x e^{x-1} + \ldots\ldots) = e^x$ for all finite real values of x. Hence

$$\text{Lim.}(1 + \frac{x}{n})^n = e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots \textit{ in infinitum.}$$

Herewith then the exponental development is established for all finite real values of the exponent.

Tulane University, May, 1896.